3.3 Differentiation Rules

Colorado National Monument

Photo by Vickie Kelly, 2003

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A quick review

If f has a derivative at $x = a$, then f is continuous at $x = a$.

Since a function must be continuous to have a derivative, if it has a derivative then it is continuous.

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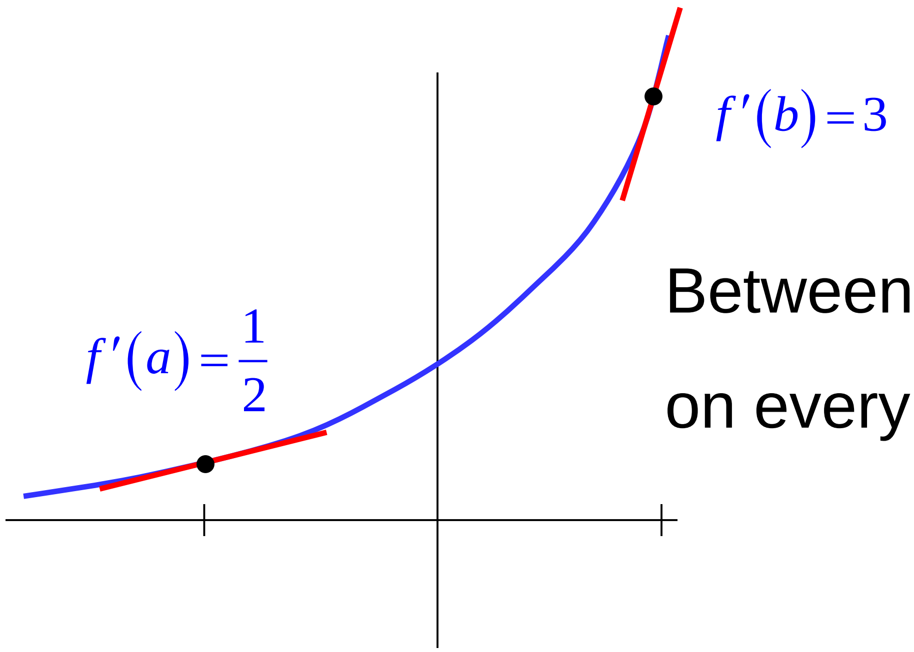
Intermediate Value Theorem for Derivatives

If a and b are any two points in an interval on which f is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$.



Between a and b, f' must take on every value between $\frac{1}{2}$ and 3.

If the derivative of a function is its slope, then for a constant function, the derivative must be zero.

$$\frac{d}{dx}(c) = 0$$

example: $y = 3$

$$y' = 0$$

The derivative of a constant is zero.

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We saw that if $y = x^2$, $y' = 2x$.

This is part of a pattern.

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

↑

power rule

examples:

$f(x) = x^4$

$f'(x) = 4x^3$

$y = x^8$

$y' = 8x^7$

→

constant multiple rule:

$$\frac{d}{dx}\left(cu\right)=c\frac{du}{dx}$$

examples:

$$\frac{d}{dx}cx^{n}=cnx^{n-1}$$

$$\frac{d}{dx}7x^{5}=7\cdot 5x^{4}=35x^{4}$$

→

constant multiple rule:

$$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

sum and difference rules:

$$\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$$

$$y = x^4 + 12x$$

$$y' = 4x^3 + 12$$

$$y = x^4 - 2x^2 + 2$$

$$\frac{dy}{dx} = 4x^3 - 4x$$

→

Find the derivative of each function below

$$y = x^3 + x + 2$$

$$y = 2x^{3/2} + x^{-2}$$

$$y' = 3x^2 + 1$$

$$\frac{dy}{dx} = 3x^{1/2} - 2x^{-3}$$

Example:

Find the horizontal tangents of: $y = x^4 - 2x^2 + 2$

$$\frac{dy}{dx} = 4x^3 - 4x$$

Horizontal tangents occur when slope = zero.

$$4x^3 - 4x = 0$$

$$x^3 - x = 0$$

$$x\left(x^2 - 1\right) = 0$$

$$x\left(x+1\right)\left(x-1\right) = 0$$

$$x = 0, -1,\ 1$$

Plugging the x values into the original equation, we get:

$$y = 2,\ y = 1,\ y = 1$$

(The function is even, so we only get two horizontal tangents.)

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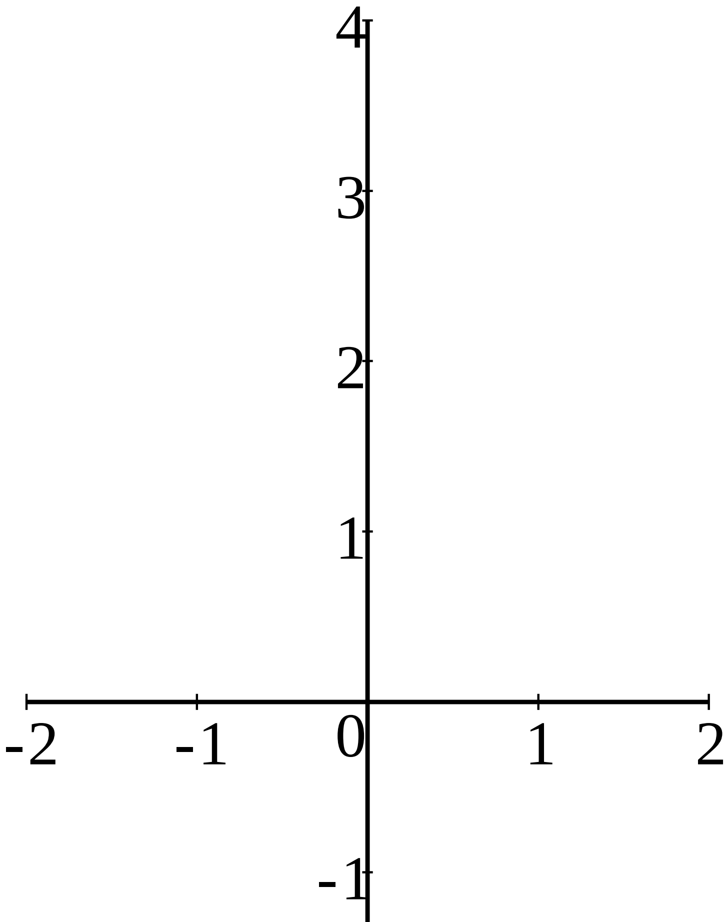
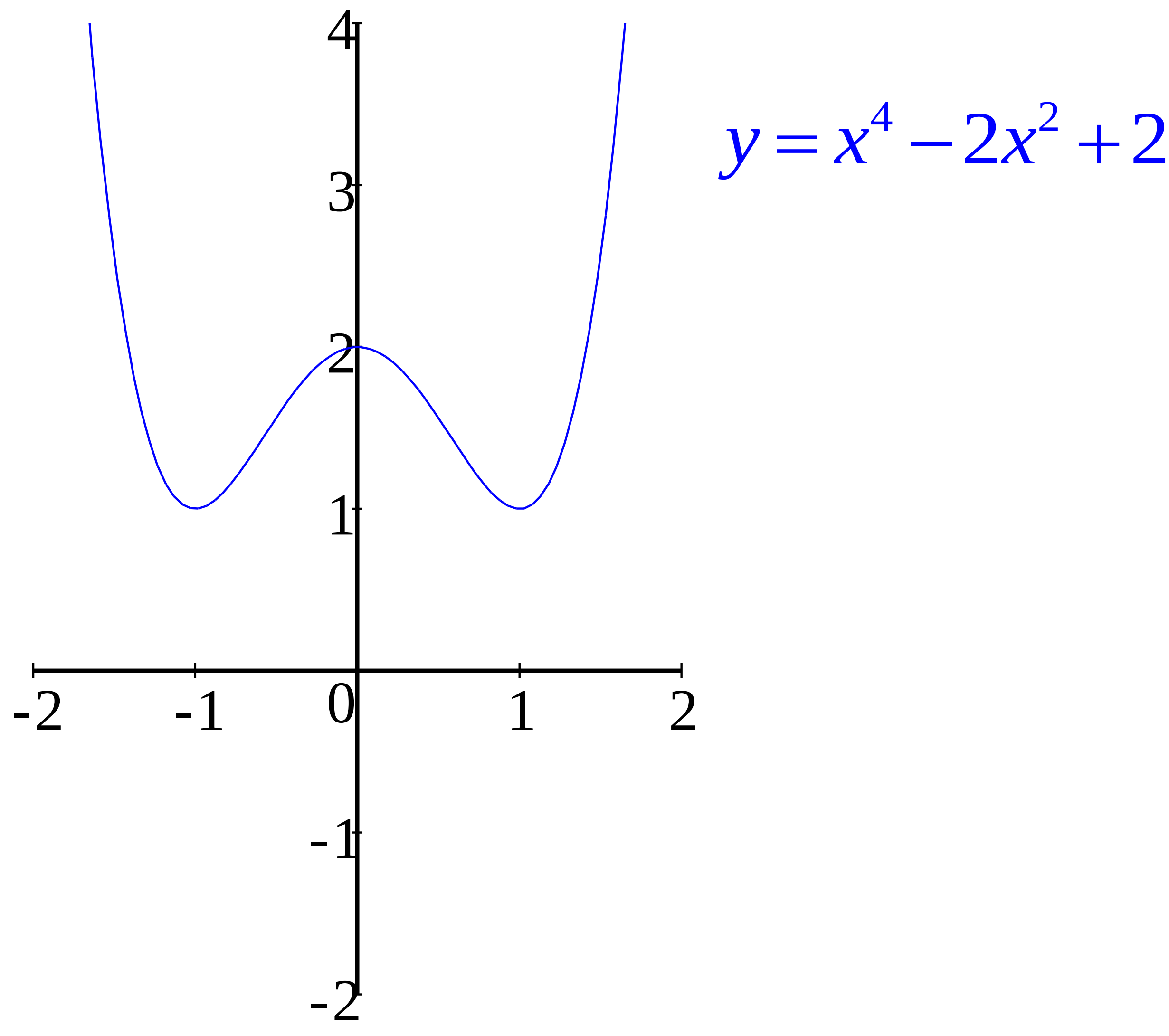
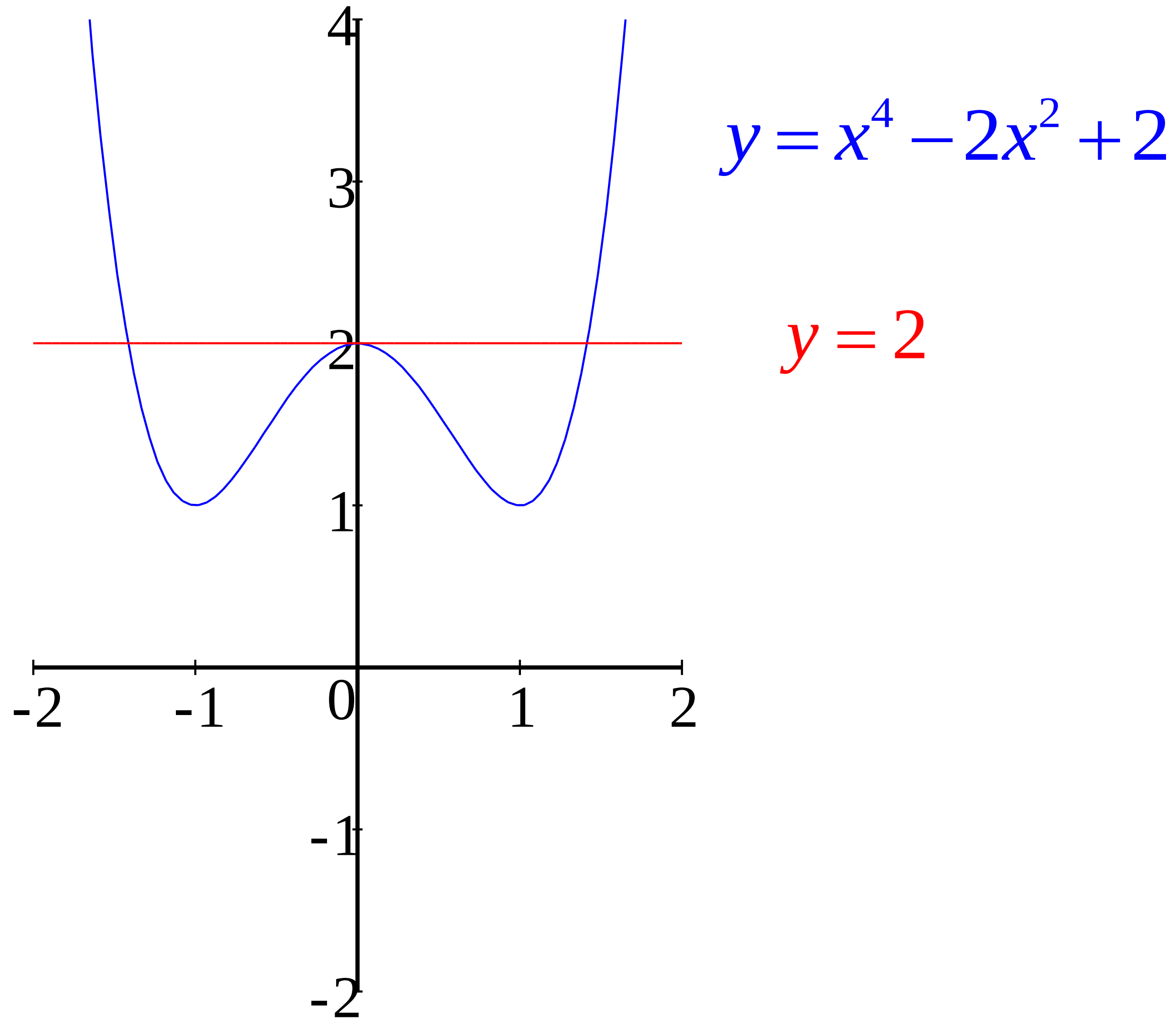
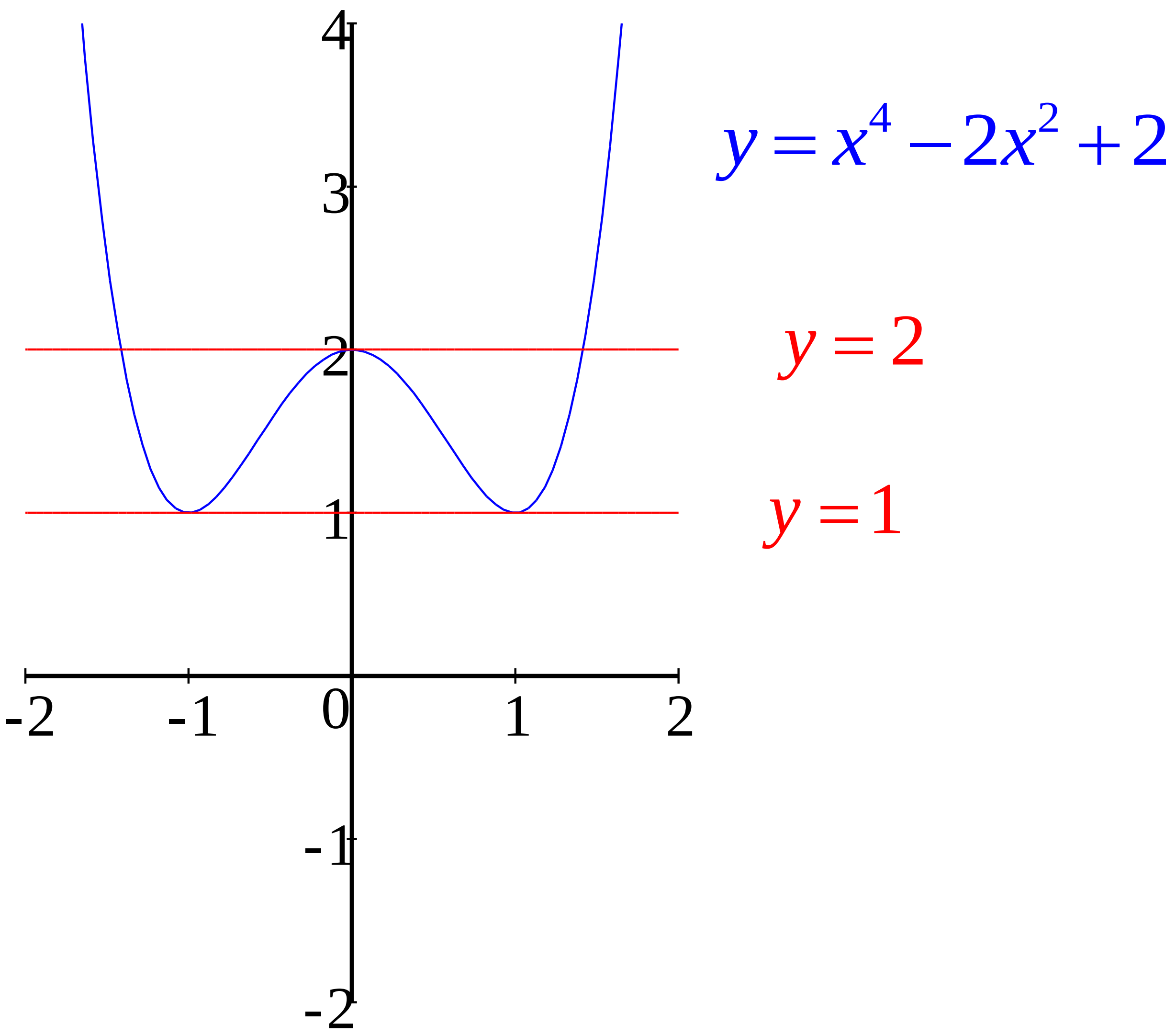

4
3
2
1
0
-1
-2
-2
-1
1
2

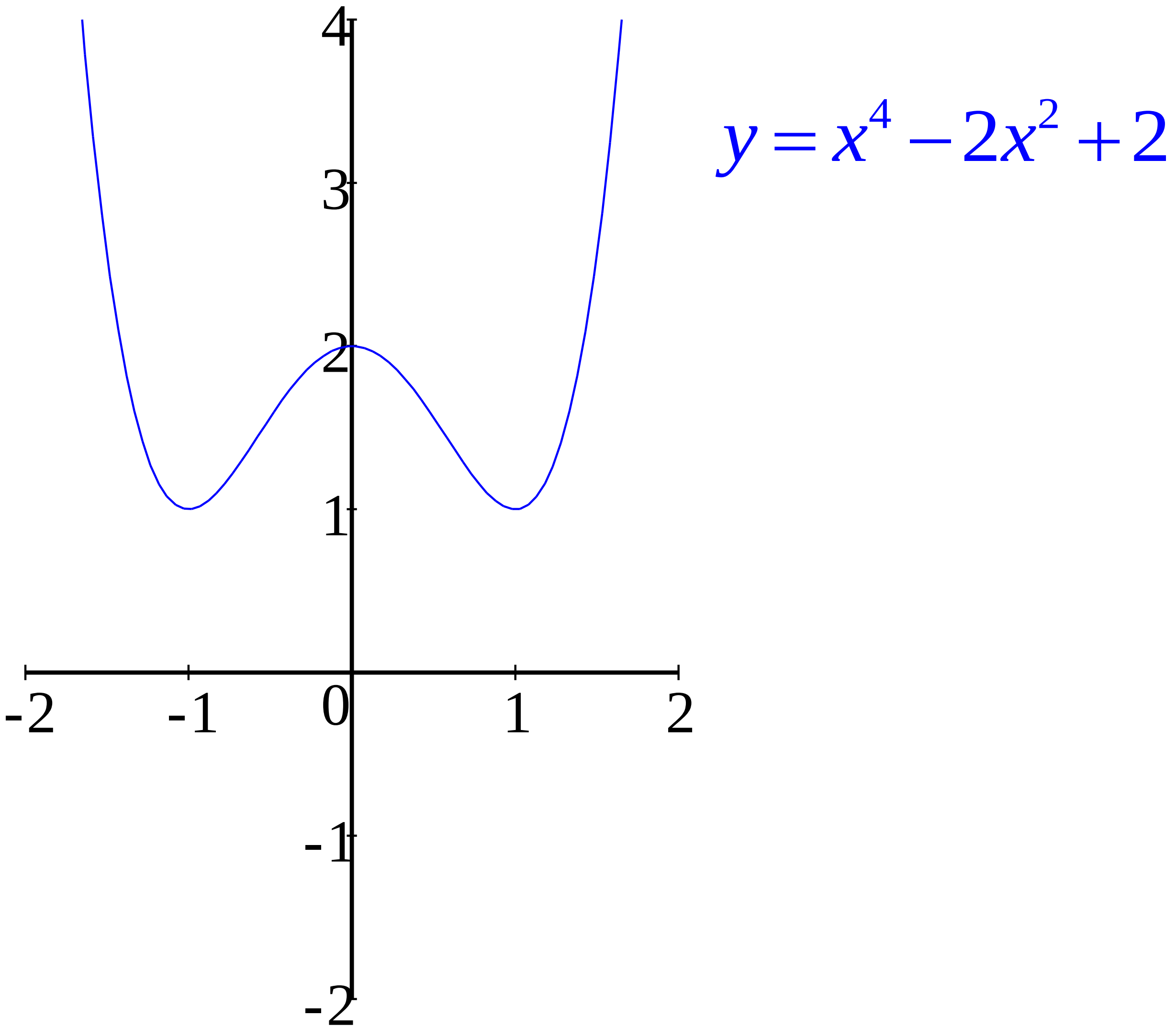

$y = x^4 - 2x^2 + 2$
4
3
2
1
-1
-2
-2
-1
0
1
2



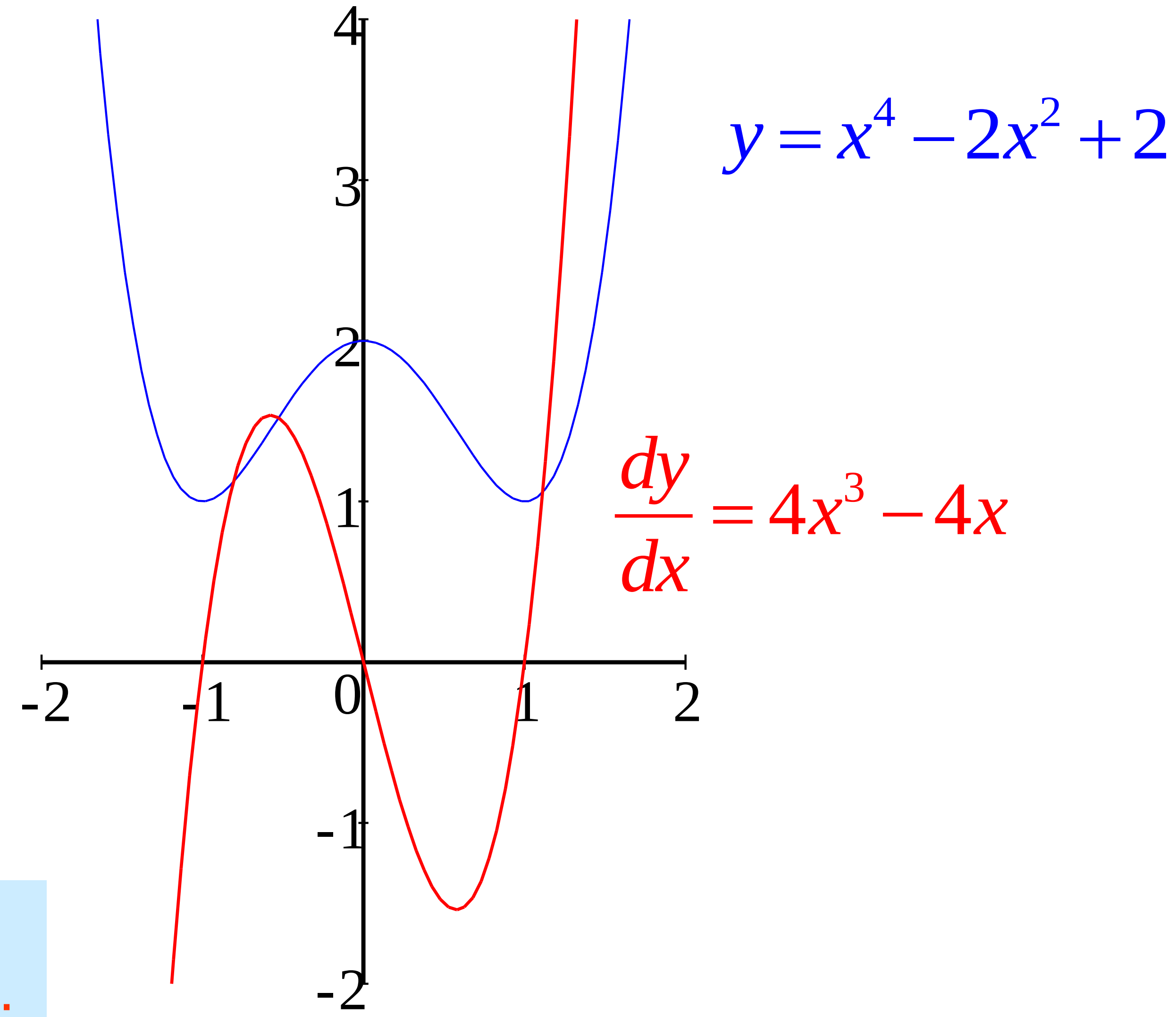
4
3
2
1
0
-1
-2
-2
-1
1
2
$y = x^4 - 2x^2 + 2$
$y = 2$



4
3
2
1
0
-1
-2
-2
-1
1
2
$y = x^4 - 2x^2 + 2$
$y = 2$
$y = 1$



$y = x^4 - 2x^2 + 2$
4
3
2
1
0
-1
-2
-2
-1
1
2



First derivative (slope) is zero at:

$$x = 0, -1, 1$$

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$$\frac{d}{dx}\left[\left(x^2+3\right)\left(2x^3+5x\right)\right]$$

$$\frac{d}{dx}\left(2x^5+5x^3+6x^3+15x\right)$$

$$\frac{d}{dx}\left(2x^5+11x^3+15x\right) = 10x^4+33x^2+15$$

→

product rule:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Notice that this is <u>not</u> just the product of two derivatives.

This is sometimes memorized as: $d(uv) = u\ dv + v\ du$

$$\frac{d}{dx}\left[\left(x^2+3\right)\left(2x^3+5x\right)\right] = \left(x^2+3\right)\left(6x^2+5\right) + \left(2x^3+5x\right)\left(2x\right)$$

$$6x^4 + 5x^2 + 18x^2 + 15 + 4x^4 + 10x^2$$

$$10x^4 + 33x^2 + 15$$

(1st)(Derivative of 2nd) + (2nd)(Derivative of 1st)

(Lefty(D-righty) + (Righty)(D-lefty)

→

quotient rule:

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

or

$$d\left(\frac{u}{v}\right)=\frac{v\ du-u\ dv}{v^2}$$

$$\frac{d}{dx}\frac{2x^3+5x}{x^2+3}=\frac{\left(x^2+3\right)\left(6x^2+5\right)-\left(2x^3+5x\right)\left(2x\right)}{\left(x^2+3\right)^2}$$

(Bottom)(Derivative of Top) – (Top)(Derivative of Bottom) all divided by the bottom squared

(Low)(D-high) – (High)(D-low) all divided by low squared

→

Higher Order Derivatives:

$$y' = \frac{dy}{dx}$$ is the first derivative of y with respect to x.

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\frac{dy}{dx} = \frac{d^2 y}{dx^2}$$ is the <u>second</u> derivative.

(y double prime)

$$y''' = \frac{dy''}{dx}$$ is the third derivative.

$$y^{(4)} = \frac{d}{dx}y'''$$ is the fourth derivative.

We will learn later what these higher order derivatives are used for.

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Product Rule

(1st)(Derivative of 2nd) + (2nd)(Derivative of 1st)

(Lefty)(D-righty) + (Righty)(D-lefty)

Quotient Rule

(Bottom)(Derivative of Top) – (Top)(Derivative of Bottom) all divided by the bottom squared

(Low)(D-high) – (High)(D-low) all divided by low squared

24. Suppose u and v are differentiable functions at x = 2

$$u(2) = 3, \ u'(2) = -4, \quad v(2) = 1, \quad v'(2) = 2$$

Find

$$a)\ \frac{d}{dx}(uv) \qquad b)\ \frac{d}{dx}\left(\frac{u}{v}\right) \qquad c)\ \frac{d}{dx}\left(\frac{v}{u}\right)$$

$$d)\ \frac{d}{dx}(3u - 2v + 2uv)$$