Differentiable $\Rightarrow$ Instantaneous $=$ Avg Rate

Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time $t$, $0 \leq t \leq 6$, is given by a differentiable function $C$, where $t$ is measured in minutes. Selected values of $C(t)$, measured in ounces, are given in the table.

<table>
<thead>
<tr>
<th>$t$ (minutes)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C(t)$ ounces</td>
<td>0</td>
<td>5.3</td>
<td>8.8</td>
<td>11.2</td>
<td>12.8</td>
<td>13.8</td>
<td>14.2</td>
</tr>
</tbody>
</table>

Is there a time $t$, $3 \leq t \leq 6$, at which $C'(t) = 1$. Justify your answer.

\[ \frac{14.2-11.2}{6-3} = \frac{1}{3} \]

Instantaneous Rate

\[ \text{Avg Rate} = \text{Instantaneous Rate} \]

Yes. Because $C(t)$ is differentiable and the Instantaneous rate of change is the Avg rate of change from $3 \leq t \leq 6$

Let $g$ be a continuous function with $g(2) = 5$. The graph of the piecewise-linear function $g'$, the derivative of $g$, is shown for $-3 \leq x \leq 7$.

\[ \text{ Avg Rate } = \frac{1 - (-4)}{1 - (-3)} = \frac{5}{4} \]

\[ \text{ Instantaneous Rate } \]

\[ \text{ No. Because } g'(x) \text{ is not differentiable at } x = -1 \]

Find the average rate of change of $g(x)$, on the interval $-3 \leq x \leq 1$. Does the Mean Value Theorem applied on the interval $-3 \leq x \leq 1$ guarantee a value of $c$, for $-3 < c < 1$, such that $g'(c)$ is equal to this average rate of change? Why or why not?
A car is traveling on a straight road. For $8 \leq t \leq 24$ seconds, the car’s velocity $v(t)$, in meters per second, is modeled by the piecewise-linear function defined by the graph.

\[ 
\text{No, because } v(t) \text{ is not differentiable at } t = 4. 
\]

Find the average rate of change of $v$ over the interval $0 \leq t \leq 16$. Does the Mean Value guarantee a value of $c$, for $0 < c < 16$, such that $v'(c)$ is equal to this average rate of change? Why or why not?

2004 BCB3

A test plane flies in a straight line with positive velocity $v(t)$, in miles per minute at time $t$ minutes, where $v$ is a differentiable function of $t$. Selected values of $v(t)$ are shown.

<table>
<thead>
<tr>
<th>$t$ (min)</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v(t)$ (npm)</td>
<td>7</td>
<td>9.2</td>
<td>9.5</td>
<td>9.2</td>
<td>4.5</td>
<td>2.4</td>
<td>4.5</td>
<td>4.9</td>
<td>7.3</td>
</tr>
</tbody>
</table>

Based on the values in the table, what is the smallest number of instances at which the acceleration of the plane could equal zero on the open interval $0 < t < 40$? Justify your answer.

Two. Since $v(t)$ is differentiable, the average rate of change $= 0$ when the values of $v(t)$ are the same. This happens between $5 \leq t \leq 15$ and $20 \leq t \leq 30$. 

2009 BC3
A continuous function $f$ is defined on the closed interval $-4 \leq x \leq 6$. The graph of $f$ consists of a line segment and a curve that is tangent to the x-axis at $x = 3$, as shown in the figure above. On the interval $0 < x < 6$, the function $f$ is twice differentiable, with $f''(x) > 0$.

Is there a value $a$, for which the Mean Value Theorem, applied to the interval $[a, 6]$, guarantees a value $c$, $a < c < 6$, at which $f'(c) = \frac{-1}{6}$? Justify your answer.

Yes at $a = 0$. Since $f$ is differentiable from $0 \leq x \leq 6$ and the avg rate of change = instantaneous rate of change.

2011 BCB5

Ben rides a unicycle back and forth along a straight east-west track. The twice-differentiable function $B$ models Ben’s position of the track, measured in meters from the western end of the track, at time $t$, measured in seconds from the start of the ride. The table gives values of $B(t)$ and Ben’s velocity, $v(t)$, measured in meters per second, at selected times $t$.

<table>
<thead>
<tr>
<th>$t$ (seconds)</th>
<th>0</th>
<th>15</th>
<th>40</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B(t)$ (meters)</td>
<td>100</td>
<td>136</td>
<td>9</td>
<td>46</td>
</tr>
<tr>
<td>$v(t)$ (meters per second)</td>
<td>2</td>
<td>1.5</td>
<td>2.5</td>
<td>4.6</td>
</tr>
</tbody>
</table>

For $15 \leq t \leq 60$, must there be a time $t$ when Ben’s velocity is -2 meters per second? Justify your answer.

$\text{AvgRate} = \frac{136 - 46}{15 - 60} = -2 \text{ instantaneous}$

2. Let $f$ be the function defined by $f(x) = x + \ln(x)$. What is the value of $c$ for
Why L’Hopital’s Works

Sketch the graph of two curves with the following characteristic \( f(2) = g(2) = 0 \).

a) Write the tangent line for \( f(x) \)
\[
(2,0) + f'(x)(x-2) = 0 + f'(2)(x-2)
\]

b) Write the tangent line for \( g(x) \)
\[
(2,0) + g'(x)(x-2) = 0 + g'(2)(x-2)
\]

c) \[
\lim_{x \to 2} \frac{f(x)}{g(x)} = \frac{f(2)}{g(2)} = \frac{0}{0}
\]

\[
\lim_{x \to 2} \frac{f'(x)(x-2)}{g'(x)(x-2)} = \lim_{x \to 2} \frac{f'(2)}{g'(2)}
\]

i) \[
\lim_{x \to 0} \frac{2x^2}{x^2} = \frac{0}{0}
\]

\[
\lim_{x \to 0} \frac{4x}{2x} = \frac{0}{0}
\]

\[
\lim_{x \to 0} \frac{4}{2} = 2
\]

ii) \[
\lim_{x \to 0} \frac{\sin(5x)}{x} = \frac{0}{0}
\]

\[
\lim_{x \to 0} \frac{\cos(5x) \cdot 5}{1} = 5
\]
4) \( \lim_{x \to 0} \frac{\sqrt[3]{x} - 1}{x - 1} = \frac{0}{0} \)

\( \lim_{x \to 1} \frac{x^{\frac{1}{3}} - 1}{x - 1} \)

\( \lim_{x \to 1} \frac{\frac{1}{3} x^{-\frac{2}{3}}}{\frac{1}{1}} = \frac{1}{3} \)

A) \( \lim_{x \to \infty} \frac{x^3 - 1}{4x^3 - x - 3} = \frac{1}{4} \)

49) \( \lim_{x \to 0} \frac{x^3 - 1}{4x^3 - x - 3} = \frac{0}{0} \)

\( \lim_{x \to 1} \frac{3x^2}{12x^2 - 1} = \frac{3}{11} \)

27) \( \lim_{x \to 0} \frac{\ln(x)}{x} = \frac{0}{0} \)

\( \lim_{x \to 0} \frac{\ln(x)}{1} = \frac{\ln(0)}{1} = 0 \)

Indeterminate Form

35) \( \lim_{x \to \infty} \frac{\log_2(x)}{\log_3(x + 3)} = \frac{\infty}{\infty} \)

Rewrite

\( \lim_{x \to \infty} \frac{\frac{1}{x \ln 2}}{\frac{1}{(x + 3) \ln 3}} \)

\( \lim_{x \to \infty} \frac{\ln 3 \ln 2}{x \ln 2} \)

33) \( \lim_{x \to 0} \frac{\sin(x^2)}{x} = \frac{0}{0} \)

\( \lim_{x \to 0} \frac{\cos(x) \cdot 2x}{1} = \frac{0}{1} = 0 \)

Horizontal Asymptote: 0