

What you'll Learn About
 How to find the error of a series that does not alternate

Lagrange Error Bound/Taylor's Inequality/Remainder Estimation Theorem

1. Give the first term of the series for $f(x) = e^x$ centered at $x = 0$

$$P_1(x) = 1$$

2. Find the approximation for $P(1) = 1$

$$(3) \text{ Find } f(1) = e^{-1} = 1.105170918$$

4. How accurate is the approximation. $|f(x) - P_1(x)| = .105170918$

5. What is the value of the next term of the polynomial at $x = 1$

$$\text{next term} = x \rightarrow .1$$

1. Give the first two terms of the series for $f(x) = e^x$ centered at $x = 0$

$$1 + x$$

2. Find the approximation for $P(1) = 1 + .1 = 1.1$

$$(3) \text{ Find } f(1) = 1.105170918$$

4. How accurate is the approximation. $|f(1) - P_1(1)| = .005170918$

5. What is the value of the next term of the polynomial at $x = 1$

$$\text{next term} = \frac{x^2}{2!} \rightarrow \frac{1^2}{2} = .005$$

1. Give the first three terms of the series for $f(x) = e^x$ centered at $x = 0$

$$1 + x + \frac{x^2}{2}$$

2. Find the approximation for $P(1) = 1 + .1 + \frac{.1^2}{2} = 1.105$

$$(3) \text{ Find } f(1) = 1.105170918$$

4. How accurate is the approximation. $|f(1) - P_2(1)| = .000170918$

5. What is the value of the next term of the polynomial at $x = 1$

$$f^{(3)}(0) = 1$$

$$25 | \text{Page } \frac{f^{(3)}(0)x^3}{3!} = \text{next term} = \frac{1x^3}{3!} \rightarrow \frac{.1^3}{6} = \frac{.001}{6} = .000166666$$

$$|f(x) - P(x)| \leq R$$

Where

$$R =$$

(Max of the next derivative on the given interval) $(x-c)^{n+1}$

$$(n+1)!$$

Where $x-c$ is the distance from the center

Where n is the order

We must build the next term a little bit bigger to have a good boundary for the error.

Remember, whenever you see this, $|f(x) - P(x)| \leq R$, you are finding error bound

whenever you see this, $|f(x) - P(x)|$, you are finding the actual error between the function and the approximation from the polynomial

Actual difference is less than

1. Give the first 4 terms of the series for $f(x) = e^x$ centered at $x = 0$

$$1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

2. Use Taylors Inequality to determine the error bound $|f(x) - P(x)| \leq R$

$$f(x) = e^x$$

$$f'(x) = e^x$$

$$f''(x) = e^x$$

$$f'''(x) = e^x$$

$$f^{(4)}(x) = e^x$$

Build next term at

$$x = 0$$

$$\frac{f^{(4)}(0) x^4}{4!}$$

$$\frac{x^4}{4!}$$

$$x = .1$$

$$\frac{f^{(4)}(.1) x^4}{4!}$$

$$\frac{e^{.1} x^4}{4!} = \frac{1.105 x^4}{4!}$$

1. Find the 3rd order polynomial of the series for $f(x) = \frac{1}{(1-x)^2}$ centered at $x = 0$

$$f(x) = \frac{1}{(1-x)^2} = (1-x)^{-2}$$

$$f'(x) = +2(1-x)^{-3}$$

$$f''(x) = +6(1-x)^{-4}$$

$$f'''(x) = 24(1-x)^{-5}$$

$$f(0) = 1$$

$$f'(0) = 2$$

$$f''(0) = 6$$

$$f'''(0) = 24$$

$$P_3(x) = 1 + 2x + \frac{6x^2}{2!} + \frac{24x^3}{3!}$$

$$f^{(4)}(x) = 120(1-x)^{-6}$$

Actual Difference

2. Find the Lagrange error bound $|f(x) - P(x)| \leq R$ or the series between $0 \leq x \leq .2$

Build next term

at

$$x = 0$$

$$\frac{120x^4}{4!}$$

$$\frac{f^{(4)}(0) x^4}{4!} = \frac{120 x^4}{4!}$$

$$\frac{120(.8)^{-6} x^4}{4!} = \frac{120 x^4}{4!}$$

distance from center

26 | Page or equal to $R = .030517 \dots$

$$R = \frac{120}{4!} (.2)^4$$

4) Write the 2nd order Taylor Polynomial for $f(x) = \cos x$ at $x = \frac{\pi}{4}$.
Then use Taylor's Inequality to determine the error bound at $x = 42^\circ$

5) Write the 1st degree Taylor Polynomial for $f(x) = \arcsin x$ at $x = 0$.
Then use Taylor's Inequality to determine the error bound at $x = .2$

$$f(x) = \arcsin x \quad f(0) = 0$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2} \quad f'(0) = 1$$

$$P_1 = \frac{0x^0}{0!} + \frac{1x^1}{1!}$$

$$P_1(x) = x$$

Next Term

$$\frac{f''(0)x^2}{2!}$$

$$f''(x) = f''(x) = -\frac{1}{2}(1-x^2)^{-3/2} \cdot (-2x)$$

Build next term at

$$= \frac{x}{(1-x^2)^{3/2}}$$

$$x=0$$

$$\frac{0x^2}{2!}$$

$$x=.2$$

$$\frac{\frac{.2}{(1-.2^2)^{3/2}} x^2}{2!}$$

$$|f(x) - P(x)| \leq R$$

R = Error Bound

$$R = \frac{\frac{.2}{(1-.2^2)^{3/2}} (.2)^2}{2!}$$

.05 away
from center
on both sides
of center

9) Given that $P_1(x) = x$ represents the first order polynomial for $\sin x$ centered at $x = 0$. Use the Lagrange Error Bound to find the error when $|x| \leq .05$

$$\begin{aligned} f(x) &= \sin x \\ f'(x) &= \cos x \\ f''(x) &= -\sin x \end{aligned}$$

$$f'''(x) = -\cos x$$

Build next term
at

$$x = -.05$$

$$x = 0$$

$$x = .05$$

$$\begin{aligned} &\frac{-\cos(-.05)x^3}{3!} \\ &\frac{-.998x^3}{3!} \end{aligned}$$

$$\begin{aligned} &\frac{f'''(0)x^3}{3!} \\ &\boxed{\frac{-1x^3}{3!}} \end{aligned}$$

$$\begin{aligned} &\frac{-\cos(.05)x^3}{3!} \\ &\frac{-.998x^3}{3!} \end{aligned}$$

$$\text{error bound} \leq \left| \frac{-(.05)^3}{3!} \right|$$

14) Given that $P_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$ represents the third order Taylor polynomial for $\ln(x)$ centered at $x = 1$. Use the Lagrange Error Bound to find the error when $|x-1| \leq .1$

Summary of Error Bound

For an Alternating Series – Use the next term

For a series that is Not Alternating

1. Write down the formula for the next derivative.
2. Find the value of the next derivative at the ends of the interval and the center.
3. Whichever value is bigger is the value you use to build your error bound term

